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Partial Isometries Closed under Multiplication on Hilbert Spaces

IVAN ERDELYI

Kansas State University, Manhattan, Kansas 66502

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INTRODUCTION

The partial isometries on Hilbert spaces behave very poorly from the operator algebra standpoint. Addition and multiplication, when defined for them, do not generate new partial isometries, in general. Murray and von Neumann [1] showed that the sum extended to a countable number of partial isometries, with mutually orthogonal initial spaces M_i and with mutually orthogonal final spaces N_i, operates isometrically on the union of M_i onto the union of N_i. In the finite case, supplementary conditions requiring that the product of two partial isometries should belong to the same class, were given in [2-4].

This paper is concerned with the transmission of partial isometry through multiplication.

The notations are all standard, as R for the range, $*$ for the adjoint, $\oplus$ and $\ominus$ for direct operations, and ordinary round brackets for inner products. Furthermore, let P_L denote an orthogonal projection on a subspace L, and I be the identity operator on a Hilbert space H. A limited use is made of the generalized inverse concept for bounded linear operators (see e.g. [5, 6]), for which the superscript "$+$" seems to be widely adopted.

Some elementary properties of partial isometries (see e.g. [1, 7, 8]) will be useful in our discussion.

In a Hilbert space H, a bounded linear operator V with closed range $R(V)$ is a partial isometry, with the initial space $R(V^*)$ and final space $V \cdot R(V^*) = R(V)$, (and then V^* is also a partial isometry with the initial and final spaces interchanged) in each of the following cases:

(a) $\begin{cases} \| Vx \| = \| x \|, & x \in R(V^*) \\ Vx = 0, & x \in H \ominus R(V^*); \end{cases}$

(b) $V^*V = P_{R(V^*)}$

(c) $VV^* = P_{R(V)}$

(d) $V = VV^*V$

(e) $V^* = V^*VV^*$

(f) $V^* = V^+$.

Multiplicative Properties

Theorem 1. *Let U and V be partial isometries on a Hilbert space H, and*

$$W = UV. \tag{1}$$

The following statements are equivalent:

(i) *W is a partial isometry;*

(ii) *the initial space of U is invariant under $P_{R(V)}$, i.e.,*

$$VV^* \cdot R(U^*) \subset R(U^*); \tag{2}$$

(iii) *the range of V is invariant under $P_{R(U^*)}$, i.e.,*

$$U^*U \cdot R(V) \subset R(V); \tag{3}$$

(iv) *the operator product $P_{R(U^*)} \cdot P_{R(V)}$ is idempotent.*

Proof. (i) $\Rightarrow$ (ii): If inclusion (2) does not hold, there exists a nonzero

$$x \in R(W^*) = V^* \cdot R(U^*)$$

such that

$$y = Vx \notin R(U^*).$$

In this case, isometry with respect to U, no longer holds

$$(Wx, Wx) = (Uy, Uy) < (y, y) = (x, x)$$

and this contradicts the hypothesis.

(i) $\Rightarrow$ (iv): If W is a partial isometry, then

$$W = WW^*W$$

and hence

$$UV = UVV^*U^*UV.$$

Property (d), applied to U and V, dilates foregoing relation as follows:

$$U \cdot U^*UVV^* \cdot V = U(U^*UVV^*)^2\, V.$$

First, we pre- and post-multiply foregoing relation by U^* and V^*, respectively,

$$U^*U \cdot U^*UVV^* \cdot VV^* = U^*U(U^*UVV^*)^2\, VV^*,$$

next, we develop the square in the right-hand side,

$$U^*U \cdot U^*UVV^* \cdot VV^* = U^*U \cdot U^*UVV^* \cdot U^*UVV^* \cdot VV^*$$

and finally, we condense the products of projections,

$$U^*UVV^* = (U^*UVV^*)^2,$$

thus

$$P_{R(U^*)} \cdot P_{R(V)} = (P_{R(U^*)} \cdot P_{R(V)})^2.$$

(ii) $\Rightarrow$ (i): If (2) holds U^*U is a projection on $V \cdot R(W^*)$, in particular. Hence, for any $x \in R(W^*)$, we have successively:

$$(Wx, Wx) = (U^*U(Vx), Vx) = (Vx, Vx) = (x, x).$$

(iv) $\Rightarrow$ (i): Assuming the idempotence of the product $P_{R(U^*)} \cdot P_{R(V)}$, we have successively:

$$\begin{aligned} WW^*W &= UVV^*U^*UV = U \cdot U^*UVV^* \cdot U^*UVV^* \cdot V \\ &= U(U^*UVV^*)^2 V = U \cdot U^*UVV^* \cdot V = UV = W, \end{aligned}$$

and by property (d), W is a partial isometry.

(i), (ii) $\Leftrightarrow$ (iii): Inclusion (2) applied to the adjoint $W^* = V^*U^*$ of the product (1) gives (3).

In the particular cases, when U and V are partial isometries, and in addition, either U is an isometry or V a co-isometry, (adjoint of an isometry), conditions (ii), (iii), (iv) of Theorem 1 are automatically satisfied, thus $W = UV$ is a partial isometry. Moreover, if U is an isometry, the initial space of W coincides with the initial space of V, and similarly, when V is a co-isometry, the final space of W is the same with the final space of U.

The transmission of partial isometry through the product of two partial isometries may be extended to the product of a finite number of partial isometries, by the following:

THEOREM 2. *Let $V_1, V_2, \ldots, V_n$ be n partial isometries on a Hilbert space H, and*

$$W_i = V_1V_2 \cdots V_i, \qquad i = 1, 2, \ldots, n. \tag{4}$$

The following statements are equivalent:

(i) *$W_2, W_3, \ldots, W_n$ are partial isometries;*

(ii) *the carrier of W_{i-1} is invariant under $P_{R(V_i)}$, i.e.,*

$$V_iV_i^* \cdot R(W_{i-1}^*) \subset R(W_{i-1}^*), \qquad \text{for} \qquad i = 2, 3, \ldots, n;$$

(iii) *the range of V_{i+1} is invariant under $P_{R(W_i^*)}$, i.e.,*

$$W_i^*W_i \cdot R(V_{i+1}) \subset R(V_{i+1}), \qquad \text{for} \qquad i = 1, 2, \ldots, n-1;$$

(iv) *the operator products* $P_{R(W_i^*)} \cdot P_{R(V_{i+1})}$, $i = 1, 2, \ldots, n-1$ *are idempotent.*

PROOF. For $n = 2$, it is reduced to Theorem 1. Assume that (i), (ii), (iii), (iv) are equivalent for $n = m$, and consider the product

$$W_{m+1} = W_m V_{m+1} ,$$

where V_{m+1} is a partial isometry. Then, by Theorem 1, W_{m+1} is a partial isometry iff (= if and only if) any of the following conditions holds:

$$V_{m+1} V_{m+1}^* \cdot R(W_m^*) \subset R(W_m^*),$$

$$W_m^* W_m \cdot R(V_{m+1}) \subset R(V_{m+1}),$$

$$P_{R(W_m^*)} \cdot P_{R(V_{m+1})} \text{ is idempotent.}$$

Thus, by induction, the proof is complete.

It is well known that generally the "reverse order law" for the inverse of operator products, i.e., $(AB)^{-1} = B^{-1}A^{-1}$ is not inherited by the generalized inverse. When it is (e.g., [9-11]), then we have:

THEOREM 3. *The product W_n of n partial isometries $V_1, V_2, \ldots, V_n$ is a partial isometry iff*

$$(V_1 V_2 \cdots V_n)^+ = V_n^+ V_{n-1}^+ \cdots V_1^+.$$

PROOF. If:

$$W_n^+ = V_n^+ V_{n-1}^+ \cdots V_1^+ = V_n^* V_{n-1}^* \cdots V_1^* = W_n^*$$

Only if:

$$W_n^+ = W_n^* = V_n^* V_{n-1}^* \cdots V_1^* = V_n^+ V_{n-1}^+ \cdots V_1^+.$$

THEOREM 4. *Let U and V be partial isometries on a Hilbert space H. The product $W = UV$ is a partial isometry in each of the following cases:*

(i) *U and VV^* commute, $U \cdot VV^* = VV^* \cdot U$;*

(ii) *V and U^*U commute, $V \cdot U^*U = U^*U \cdot V$.*

PROOF. We confine our proof to case (i):

$$\begin{aligned} WW^*W &= U \cdot VV^* \cdot U^*UV = VV^* \cdot UU^*U \cdot V = VV^* \cdot UV \\ &= U \cdot VV^*V = UV = W. \end{aligned}$$

If A and B are contractions, i.e., $\| A \| \leqslant 1$, $\| B \| \leqslant 1$, in a Hilbert space H, then the operator matrices

$$M(A) = \begin{bmatrix} A & (I - AA^*)^{1/2} \\ 0 & 0 \end{bmatrix}, \qquad M(B) = \begin{bmatrix} B & (I - BB^*)^{1/2} \\ 0 & 0 \end{bmatrix}$$

are partial isometries on $H \oplus H$, [7, 8].

Moreover, we have

THEOREM 5. *The product*

$$M = M(A) \cdot M(B) \tag{5}$$

is a partial isometry iff A is a partial isometry.

PROOF. A direct computation shows that if AA^* is a projection then so is

$$MM^* = \begin{bmatrix} AA^* & 0 \\ 0 & 0 \end{bmatrix}.$$

Conversely, suppose that the product (5) is a partial isometry. Then, after some elementary operations, we obtain

$$M^* = \begin{bmatrix} B^*A^* & 0 \\ (I - BB^*)^{1/2} A^* & 0 \end{bmatrix},$$

$$M^*MM^* = \begin{bmatrix} B^*A^*AA^* & 0 \\ (I - BB^*)^{1/2} A^*AA^* & 0 \end{bmatrix};$$

and since

$$M^* = M^*MM^*$$

we have

$$B^*A^* = B^*A^*AA^* \tag{6}$$

$$(I - BB^*)^{1/2} A^* = (I - BB^*)^{1/2} A^*AA^*. \tag{7}$$

If we premultiply (7) by $(I - BB^*)^{1/2}$ and use (6), we obtain

$$A^* = A^*AA^*,$$

and the proof is complete.

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